Lattice (Binomial Trees) Version 1.2

1 Introduction

This plug-in implements different binomial trees approximations for pricing contingent claims and allows Fairmat to use some of the most popular binomial trees approximations to price a subset of the models which is possible to use within Fairmat.

2 How to use the plug-in

When the Lattice plug-in is installed, the following options appear in the Fairmat user interface:

- Under the Settings menu, a new item which will allow you to choose between **Simulation** and **Lattice** appears;
- Under the Settings menu, the "Numerical Settings" dialog window (See Figure 1) contains a new tab which allows to specify the lattice specific parameters: between them, the "approximation methods" which will be used to build the approximation of the dynamics. The supported discretization schemes are:
  - "CRR/BEG";
  - "NEK (Ekval)";
  - "GLT";
  - "AGLT".
- A new process called custom dynamic will be available on the stochastic processes list. Custom dynamic allows you to specify the outcomes of your stochastic process, and it is very useful when no statistical information is available.

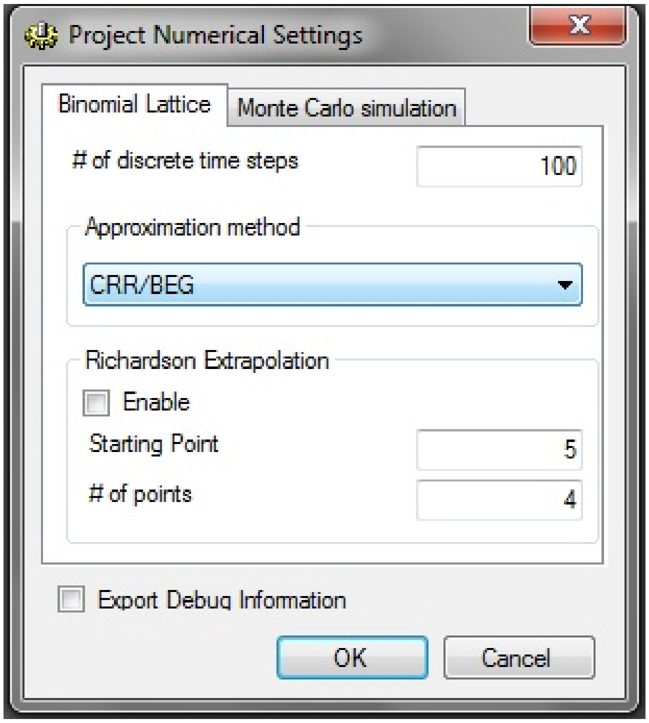


Figure 1: Binomial Trees Numerical Settings Form

2.1 How to use the custom dynamic

Note: Custom dynamic is experimental and may be subject to changes! The **Custom dynamic** stochastic process generates a binomail trees where the trees levels are determined by the user. This feature allows you to specify the outcomes of your stochastic process, and it is very useful when no statistical information is available. The following limitation arises in projects using custom dynamic:

- The actual implementation overrides the number of time steps and assumes that time step are equally distributed.
- If in a project are present more custom dynamics they must have the same number of columns.

3 Implementation details

In the binomial lattice approaches the basic assumption is that the value of each source of uncertainty can move up or down by a given amount in a small time

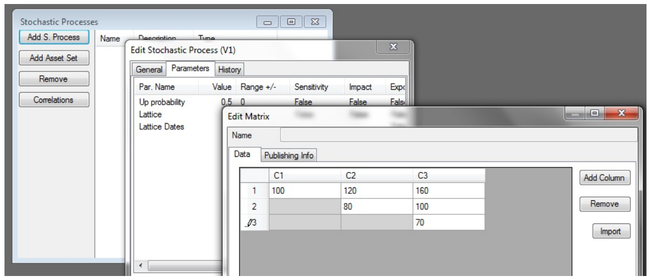


Figure 2: Example of Lattice editing with custom dynamic.

period. By combining this simple dynamic for all the sources of uncertainty we obtain a multivariate binomial tree. Then we can evaluate the project and its embedded options by backward dynamic programming at every node of the binomial tree. Fairmat® enables us to evaluate Real Options with a binomial lattice approach that allows however many underlying assets you want; the limitations are only in computing time and in memory required to calculate the results. As you see in the previous section, in the current version of Fairmat® four possible lattice schemes can be used to evaluate options:

- “CRR/BEG”: the standard Cox-Ross-Rubinstein scheme [1] (extended to multidimensional framework by Boyle-Evnine-Gibbs [2]);
- “NEK (Ekval)” the Ekvall’s approach [3].
- “GLT”/“AGLT”: a log-transformed scheme (proposed by Trigeorgis [4]) and extended to a multidimensional setting by Gamba-Trigeorgis [5].

Additionally those methods can be improved by using the Richardson Extrapolation.

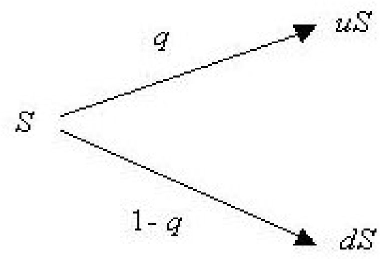
In the following subsections, the Cox-Ross-Rubistein and the Log-Transformed scheme are explained with more detail.

3.1 Cox-Ross-Rubistein scheme

Each underlying asset follows a stationary multiplicative binomial process over discrete periods described by:

where the state variable at the beginning of a given period, S, may increase (by a multiplicative factor u) with probability q to uS or decrease with complementary probability $(1-q)$ to dS at the end of a period Δt. Thus u and d represent the (continuously compounded or logarithmic) rate of return if the underlying asset moves up or down, respectively, with $d = 1/u$.

Letting r denote the riskless interest rate over we require $u > (1+r) > d$. If these inequalities did not hold, there would be profitable riskless arbitrage



opportunities involving only the asset and riskless borrowing and lending.

We must adjust the interval-dependent variables r, u, d in such a way that we obtain empirically realistic results as Δt becomes smaller, or, equivalently, as $n = (t/\Delta t) \to \infty$, where t is a fixed calendar time (for instance, the maturity of an option) and n is the number of periods of length Δt prior to the expiry.

We want to approximate the continuous-time Geometric Brownian Motion with the discrete process. If we take:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

$$q = \frac{1}{2}\Big(1 + \frac{g - \sigma^2/2}{\sigma}\sqrt{\Delta t}\Big)$$

the discrete process has the same distribution of the continuous process when $n \to \infty$ (means and variances coincide in the limit).

For the mean-reverting processes:

$$dV(t) = \eta(\overline{V} - V(t))dt + \sigma dz$$

the parameters are:

$$u = \sigma\sqrt{\Delta t} \quad d = -\sigma\sqrt{\Delta t}$$

$$p = \begin{cases} \frac{1}{2}\Big(1 + \frac{\eta\left(\overline{V} - V(t)\right)}{\sigma}\sqrt{\Delta t}\Big) & p \in]0, 1[\\ 0 & p \leq 0 \\ 1 & p \geq 1. \end{cases}$$

For the other kind of mean-reverting processes (log-MR):

$$dV(t) = \eta V(t)(\overline{V} - V(t))dt + \sigma V(t)dz$$

the parameters are:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \begin{cases} \frac{1}{2}\Big(1 + \frac{\eta\left(\overline{V} - V(t)\right) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\Big) & p \in]0,1[\\ 0 & p \leq 0 \\ 1 & p \geq 1. \end{cases}$$

For the Itô processes, the parameters that ensure the coincidence between limiting means and variances of continuous and discrete process are:

$$u = e^{b(V(t),t)\sqrt{\Delta t}}$$

$$q = \frac{1}{2}\Big(1 + \frac{a(V(t),t)}{b(V(t),t)}\sqrt{\Delta t}\Big)$$

To compute the value of an option depending on the underlying asset we use **Dynamic Programming**, a very general tool for dynamic optimization which is particularly useful when considering uncertainty. It breaks a whole sequence of decisions into just two components: the immediate decision, and a valuation function that incorporates the consequences of all subsequent decisions, starting with the position that results from the immediate decision.

The idea behind this decomposition is formally stated in Bellman's Principle of Optimality: an optimal policy has the property that, whatever the initial action, the remaining choices constitute an optimal policy with respect to the subproblem starting at the state that results from the initial actions.

The result of this decomposition is called the **Bellman equation**.

Suppose the current date is t and the state is x_t. Let us denote by $F_t(x_t)$ the outcome (the expected net present value of all the firm's cash flows) when the firm makes all decisions optimally from this point onwards.

When the firm chooses the control variables u_t, it gets an immediate profit flows $\pi_t(x_t, u_t)$. At the next period $(t+1)$, the state will be x_{t+1}. Optimal decisions thereafter will yield $F_{t+1}(x_{t+1})$. This is random from the perspective of period t, so we must take its expected value. That is what we call continuation value and it has to be discounted back to period t.

The firm will choose u_t to maximize the sum of the immediate profit and the continuation value and the result will be just the value $F_t(x_t)$. So, at any time t, the Bellman equation is:

$$F_t(x_t) = \max_{u_t}\left\{\pi_t(x_t, u_t) + \frac{1}{1+\rho}E_t[F_{t+1}(x_{t+1})]\right\}.$$

If the planning horizon is finite, the very last decision at its end has nothing following it, and can therefore be found using standard optimization methods. This solution then provides the valuation function appropriate to the penultimate decision. That, in turn, serves for the decision two stages from the end, and so on. In this way we can work backwards all the way to the initial condition.

Let us consider a simple example: an option to invest in a project.
The data of the problem are:

$V = 100$	Current value of the real asset (the project)
$I = 160$	Cost of the investment
$T = 3$	Time to option maturity
$u = 1.42$	Growth factor if the underlying asset moves up
$d = 1/u$	Decrease factor if the underlying asset moves down
$p = 0.37$	Probability of an up movement
$r = 6\%$	Continuously compounded yearly interest rate
$\delta = 9\%$	Dividend yield of the underlying asset
$\sigma = 35\%$	volatility of the underlying asset
$\Delta t = 1$	Length of each time step

The stationary multiplicative binomial process over discrete periods of the underlying asset is:

$$
\begin{array}{rrrr}
100 & 142 & 201 & 286 \\
 & 70 & 100 & 142 \\
 & & 50 & 70 \\
 & & & 35
\end{array}
$$

where:

$$
\begin{cases} V_t \times u \\ V_t \times d \end{cases}
$$

The value of the project is computed working backward, according to the Dynamic Programming approach. At the last relevant decision point, T, we can make the best choices to invest and thereby find the continuation value. The value of the investment opportunity is:

$$G_T = \max\left\{V_t - I, 0\right\}$$

Then at the decision point before that one and for each value of the underlying asset, we know the expected continuation value and therefore can optimize the current choice. The Bellman equation for a generic $t < T$ is:

$$G_t = \max\left\{V_t - I, \frac{1}{1+r}E_t[F_{t+1}]\right\}$$

that is the maximum between the payoff and the continuation value.
So the value of the option at each time step and also at $t = 0$ is:

$$\begin{array}{cccc} 5 & 15 & 44 & 126 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{array}$$

where:

$$G(V_T) = \max\left\{V_T - I, 0\right\},$$

$$G(V_t) = \max\left\{V_t - I, \frac{pG(V_t u) + (1-p)G(V_t d)}{1+r}\right\}.$$

3.2 Log-Transformed scheme

In the financial literature, several extensions of the CRR lattice approach presented above for Geometric Brownian Motions have been proposed. Boyle, Evnine and Gibbs (BEG) provide a straightforward extension of the Cox, Ross and Rubinstein approach to several underlying assets whose dynamics are GBM. As BEG acknowledge, their scheme provides positive probability if the size of the time step is small (i.e., if the number of steps is large enough). Unfortunately, if there are a number of underlying assets, the lattice will quickly become too complex if there are a large number of steps. So it may happen that for some values of the parameters (if the volatility or the number of steps used are small with respect to the risk-adjusted drift), the probabilities of the jumps in BEG's scheme (which is based on CRR's choice of probability) are negative, giving inaccurate estimates of the value of the option and making the method unstable.

A very important feature of the Cox-Ross-Rubinstein method, as illustrated above, is that it makes pedagogically clear the relationship between the no-arbitrage (or risk neutral valuation) argument and the hedging argument. Having said that, it does not need to be also the best approach from a numerical viewpoint. Several other approaches have been proposed.

Among other lattice approximations, the log-transformed approximation has been proposed by Trigeorgis (1991) to overcome some of the above flaws presented by the CRR approach.

This method can be applied only in the case of Geometric Brownian Motion.

Let's first consider the **uni-dimensional case**.

The standard Brownian Motion:

$$dY_t = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t,$$

$$Y_0 = y = \log x,$$

where

$$dx_t = \alpha x_t dt + \sigma x_t dZ_t,$$

is approximated by a discrete-time process

$$\widetilde{Y}_t = \widetilde{Y}_{j-1} + hU_j \quad \widetilde{Y}_0 = y$$

$j = 1, 2, \ldots, n$ with parameters:

$$\mu = \frac{m}{\sigma^2} = \frac{\alpha}{\sigma^2} - \frac{1}{2}$$

$$k = g = \sigma\sqrt{\Delta t}$$

$$h = \sqrt{k^2 + (k^2\mu)^2}$$

$$p = \frac{1}{2}\left(1 + \frac{k^2\mu}{h}\right).$$

Note that the up step here is $u = e^h$ and $d = 1/u$. A remarkable feature of this method is that, since $h \geq |k^2\mu|$, then $0 \leq p \leq 1$ with no need of external constraints on the parameters to make the algorithm stable. Thus a key feature of the log-transformed approach is that it allows for unconditional stability. This is so because the time unit is k instead of Δt; i.e., time is measured in units of variance. This feature makes the log-transformed approximation consistent at each step n (not just in the limit as $n \to \infty$ as in CRR):

$$E[\Delta\widetilde{Y}_t] = m\Delta t \quad Var[\Delta\widetilde{Y}_t] = \sigma^2\Delta t.$$

For this reason, unlike the CRR approach, it does not explode for small volatility and/or number of time steps.

The efficiency of the log-transformed method proves to be even more important when evaluating options with many underlying assets because memory and computational constraints do not allow for a very large number of steps.

In the multidimensional case, the correlation plays an important role in the ability to maintain a positive probability in all states (because of the presence of the correlation, the straightforward extension of Trigeorgis' log-transformed binomial approach to several underlying assets suffers with the same drawbacks as in CRR (and BEG) scheme: the probability distribution, for some values of the parameters, can have negative values).

Consider N correlated assets whose price dynamics X_i are described by the following geometric Brownian motions (under the Martingale probability):

$$\frac{dX_i}{X_i} = \alpha_i dt + \sigma_i dZ_i$$

$$E[dZ_i dZ_j] = \rho_{ij} dt$$

$i = 1, 2, \ldots, N$ where α_i is the risk-adjusted drift of the i-th asset price and $i \neq j$.

Given a derivative security with maturity T and price F whose payoff depends on the underlying assets prices, we want to estimate the risk-neutral price of the derivative security. Following the usual Black and Scholes argument, the valuation p.d.e. for F in the multidimensional case is:

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\sigma_i\sigma_j X_i X_j \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{N}\alpha_i X_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t} - rF = 0$$

with appropriate boundary conditions.

Since in general an analytic solution to this p.d.e. (for given boundary conditions) does not exist, we can obtain a numerical solution by approximating the continuous dynamics with a binomial lattice approach.

To illustrate the process more simply, we first show the two-dimensional case, N = 2. First, we take the log of the asset values: $Y_i = \log X_i$, $i = 1, 2$. The dynamics of Y_i are:

$$dY_i = \left(\alpha_i - \frac{1}{2}\sigma_i^2\right)dt + \sigma_i dZ_i$$

with $i = 1, 2$.

Given the time interval $[0, T]$ specified by the maturity of the option, we consider n subintervals of width $\Delta t = T/n$. We approximate the continuous-time process with the discrete-time one

$$(\widehat{Y}_1, \widehat{Y}_2).$$

The approximation criterion is the following: the discrete time process approximates the diffusion if the characteristic function of the first one approximates the characteristic function of the second one. This is equivalent to matching the first two moments of the distributions. The discrete process is:

$$\widehat{Y}_i(t) = \widehat{Y}_i(t-1) + h_i U_i(t),$$

$i = 1, 2$ e $t = 1, \ldots, n$ where is a bi-variate i.i.d. binomial random variable:

$$(U_1, U_2) = \begin{cases} (1,1) & p_1 \\ (1,-1) & p_2 \\ (-1,1) & p_3 \\ (-1,-1) & p_4 \end{cases}$$

and

$$\sum_{i=1}^{4} p_i = 1$$

Let:

$$\mu_i = \frac{\alpha_i}{\sigma_i^2} - \frac{1}{2}$$

$$k_i = \sigma_i \sqrt{\Delta t}$$

$$h_i = \sqrt{k_i^2 + (k_i^2 \mu_i)^2}$$

$$R_{ij} = k_i k_j / (h_i h_j)$$

$$M_i = k_i^2 \mu_i / h_i$$

$i = 1, 2$. Note that $u_i = e^h i$ and $d_i = 1/u_i$. Moreover, let

$$p_1 = p_{uu} = (1 + (R\rho + M_1 M_2) + M_1 + M_2)/4$$

$$p_2 = p_{ud} = (1 - (R\rho + M_1 M_2) + M_1 - M_2)/4$$

$$p_3 = p_{du} = (1 - (R\rho + M_1 M_2) - M_1 + M_2)/4$$

$$p_4 = p_{dd} = (1 + (R\rho + M_1 M_2) - M_1 - M_2)/4$$

where $\rho = \rho_{12}$ and $R = R_{12}$. With these parameters, the first moments of the increment of the discrete-time process match the first moments of the increment of the continuous-time process for any given time step Δt:

$$E[\Delta\widehat{Y}_i] = k_i^2 \mu_i = E[\Delta Y] = \left(\alpha_i - \frac{1}{2}\sigma_i\right)\Delta t$$

$$Var[\Delta\widehat{Y}_i] = k_i^2 = Var[\Delta Y] = \sigma_i^2 \Delta t$$

$$Cov[\Delta\widehat{Y}_1, \Delta\widehat{Y}_2] = \rho_{12}k_1k_2 = Cov[\Delta Y_1, \Delta Y_2] = \rho_{12}\sigma_1\sigma_2\Delta t.$$

Hence, the approximation of the bi-variate geometric Brownian motion:

$$\frac{dX_i}{X_i} = \alpha_i dt + \sigma_i dZ_i$$

$i = 1, 2$. Is given by the process

$$(\widehat{X}_1, \widehat{X}_2)$$

such that:

$$\widehat{X}_i(t) = \widehat{X}_i(t-1)e^{h_i U_i}$$

$t = 1, 2, \ldots, n$, $i = 1, 2$.

If the asset returns are uncorrelated, then the log-transformed probability is always strictly positive and the multidimensional extension of the log-transformed approximation would have the same features as in the case with one underlying asset.

We can change the co-ordinate system in order to have a set of uncorrelated diffusions, then we can evaluate an option written on multiple assets while preserving all the positive features of the log-transformed approach.

We change the basis of R^N, the market space generated by the N-dimensional diffusion of the asset returns (the symbol T denotes transposition), so that the price of the derivative security is dependent on an N-dimensional diffusion obtained by a change of basis such that its components y_i are uncorrelated. Note that if we change the basis of the market space the risk structure of the market does not change and hence we can employ risk-neutral valuation. We have to change the payoff function accordingly: denoting by $\Pi(Y)$ the payoff of the contingent claim, and W the matrix representing the change of basis, the expression of the payoff with respect to the new basis is:

$$\widehat{\Pi}(y) = \Pi(Wy).$$

The dynamics of the returns y can then be approximated by a suitable log-transformed binomial lattice that overcomes the previous problems. The probabilities of the improved log-transformed approximation are positive and lower that one for any parameter values.

In particular this method maintains the unconditional stability feature of the approach presented in the one-dimensional case by Trigeorgis (for details, see Gamba-Trigeorgis (2001)).

The economic rationale of the approach based on a change of basis is the following. We want to price the contingent claim with payoff $\Pi(Y)$, where Y is the return of the assets traded in the market, in a risk neutral setting. If the financial market is complete (that is, if the number of non-perfectly correlated traded assets is equal to the number of the sources of uncertainty), we can generate N portfolios with the original assets: we denote

$$w_i^T = (w_{i1}, \dots, w_{iN})$$

the i-th portfolio, $i = 1, \dots, N$, where w_{ij} is the position in the j-th asset in portfolio i.

We can see these portfolios as new synthetic assets spanning the (same) market space. Any contingent claim which is redundant with respect to the original assets is redundant also with respect to these synthetic assets. The N portfolios we generate are selected so as to have uncorrelated returns. The contingent claim to be priced is dependent on the returns of the synthetic assets and is denoted by

$$\widehat{\Pi}(y).$$

Since the risk structure of the market is unchanged (the market spanned by the synthetic assets is the same as the original one: the only thing that changes is the representation of returns), we can price the claim according to a risk-neutral approach with respect to a Martingale probability derived by the original one by a simple change of basis.

To illustrate this multidimensional binomial technique, we present the two-dimensional case.

Suppose:

$$dY_i = \alpha_i dt + \sigma_i dZ_i$$

$i = 1, 2$ where $\rho = \rho_{12}$ and

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

Since

$$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}$$

we get $\Lambda = (\lambda_i)$, a two-dimensional diagonal matrix, where

$$\lambda_{1,2} = \frac{1}{2}\Big(\sigma_1^2 + \sigma_2^2 \mp \sqrt{\sigma_1^4 + 2(1-2\rho^2)\sigma_1^2\sigma_2^2 + \sigma_2^4}\Big)$$

and

$$W = \begin{pmatrix} \left(\frac{\lambda_1}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)/(\rho c_1) & \left(\frac{\lambda_2}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)/(\rho c_2) \\ 1/c_1 & 1/c_2 \end{pmatrix}$$

where

$$c_i = \sqrt{1 + \frac{(\lambda_i - \sigma_2^2)^2}{\rho^2\sigma_1^2\sigma_2^2}}$$

The processes of the returns of the synthetic securities are

$$dy_i = A_i dt + B_{i1} b dZ_1 + B_{i2} dZ_2$$

$i = 1, 2$ where $B = (B_{ii}) = W^T b$ and $A = W^T a$. We approximate the distribution of y with a discrete distribution: given the time interval $[0, T]$, we consider n subintervals of width $\Delta t = T/n$. The discrete process is

$$(\widetilde{y}_1, \widetilde{y}_2)$$

with dynamics

$$\widetilde{y}_i(t) = \widetilde{y}_i(t-1) + l_i U_i(t)$$

$i = 1, 2$ and $t = 1, \ldots, n$ where is a bi-variate i.i.d. binomial random variable with distribution:

$$(U_1, U_2) = \begin{cases} (1,1) & p_1 \\ (1,-1) & p_2 \\ (-1,1) & p_3 \\ (-1,-1) & p_4 \end{cases}$$

By assigning the parameters

$$\kappa_i = A_i \Delta t$$
$$l_i = \sqrt{\lambda_i \Delta t + \kappa_i^2}$$
$$L_i = \kappa_i / l_i$$

$i = 1, 2$ and probability

$$p(s) = \frac{1}{4}(1 + \delta_{12}(s) L_1 L_2 + \delta_1(s) L_1 + \delta_2(s) L_2)$$

$s = 1, \ldots, 4$, for the discrete time process, we have the following:

$$E[\Delta\widetilde{y}_i] = \kappa_i = A_i \Delta t$$

$$Var[\Delta\widetilde{y}_i] = l_i^2 - \kappa_i^2 = \lambda_i \Delta t$$
$$Cov[\Delta\widetilde{y}_1 \Delta\widetilde{y}_2] = 0$$

Hence, this discrete process is consistent with the continuous process for any time step (not just in the limit).

The above case can be generalized to the N - dimensional case.

3.3 Richardson Extrapolation

Many methods of approximation depend on a positive parameter, say, h, which controls the accuracy of the method. As $h \to 0$, the approximations typically converge to the exact solution. In practice, one usually computes several approximations to a solution, corresponding to different values of the parameter h. It is then natural to try "extrapolating to the limit $h = 0$", that is constructing a linear combination of these approximations that is more accurate than either of them.

We consider an option with maturity $t = 1$ and we used a 4-point Richardson extrapolation based on a 3-dimensional binomial lattice with, for example, $n = 12, 24, 36$, and 48 time steps. In this way we can fit option values as a cubic function of $h = 1/n$. By extrapolating this function, we obtain an approximation to the value corresponding to $n = \infty$, that is $h = 0$.

In this particular case, the four different values of the parameter h are:

$$h_1 = 1/12 \quad h_2 = 1/24 \quad h_3 = 1/36 \quad h_4 = 1/48$$

We develop four Taylor's expansions around the four values of h:

$$f(h_1) = f(0) + f'(0)h_1 + \frac{1}{2}f''(0)h_1^2 + \frac{1}{6}f'''(0)h_1^3$$

$$f(h_2) = f(0) + f'(0)h_2 + \frac{1}{2}f''(0)h_2^2 + \frac{1}{6}f'''(0)h_2^3$$

$$f(h_3) = f(0) + f'(0)h_3 + \frac{1}{2}f''(0)h_3^2 + \frac{1}{6}f'''(0)h_3^3$$

$$f(h_4) = f(0) + f'(0)h_4 + \frac{1}{2}f''(0)h_4^2 + \frac{1}{6}f'''(0)h_4^3$$

In matrix notation, we have:

$$b = Ax$$

where:

$$b = \begin{bmatrix} f(h_1) \\ f(h_2) \\ f(h_3) \\ f(h_4) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & h_1 & h_1^2 & h_1^3 \\ 1 & h_2 & h_2^2 & h_2^3 \\ 1 & h_3 & h_3^2 & h_3^3 \\ 1 & h_4 & h_4^2 & h_4^3 \end{bmatrix}$$

$$x = \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \\ f'''(0) \end{bmatrix}$$

The Richardson extrapolated value is $f(0)$. We solve the problem by computing the inverse of matrix A and then considering

$$x = A^{-1}b$$

that is, $f(0)$ is obtained by multiplying the first row of the inverse matrix of A by the column vector b.

References

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