

## 1 Introduction

This plug-in implements an Economic Scenarios Generator (ESG) which is able to generate several coherent and market consistent (risk-neutral or real-world) scenarios for several asset class including:

- Zero coupon bonds
- European inflation
- Defaultable bonds / credit spreads
- Selected Equities (i.e. Euro Stoxx 50, or any other regional Index or Security): including price, total returns and dividend yield.

## 2 How to use the Simulator

### 2.1 Simulation model setup

ESG generates economic scenarios for any set of stochastic processes available in a Fairmat model, actually a Fairmat project is used to define what has to be simulated. The simplest way is to start from a template: the ESG plug-in makes available the following specific templates for Economic Scenarios Generator purposes.

- **Interest Rates / Risky Bonds\*/ Equity:** this template contains one interest rate process and one equity (in which currency and tickers can be customized).
- **Interest Rates/ Risky Bonds\*/ Inflation / Equity:** this template contains on interest rate process and one equity (in which currency and tickers can be customized).
- **Interest Rates/ Risky Bonds\*/ Inflation / Two Equities:** this template contains an example in which a basked with two equities can be simulated, but it is possible to add as many equity as needed.

Templates are available in `File/Home/Templates`

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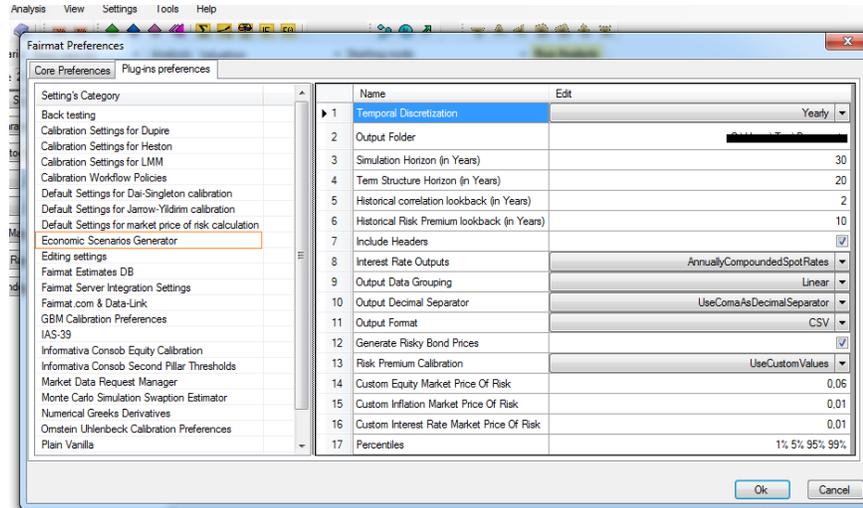


Figure 1: Fairmat ESG preferences.

### 3 Scenario Generation Setup

ESG outputs can be highly customized through the ESG preferences which can be found in **Settings/Fairmat Preferences/Plug-in Preferences/Economic Scenario Generator**

Currently the plug-in supports the following options:

- **Temporal Discretization:** it defines the time step with which the scenarios will be generated. Choose between **Weekly**, **Monthly** or **Yearly**.
- **Output Folder:** it defines the path in the user's computer where the scenarios will be generated. The different CSV files are stored inside an automatically generated folder in the path specified by this setting.
- **Simulation Horizon (in Years):** it defines the number of years the user wishes to project into the future the models.
- **Term Structure Horizon (in Years):** it defines the number of years the user wishes to take into consideration for his/her term structure of discount factors.
- **Historical correlation lookback (in Years):** it defines the number of years the user wishes to take into consideration to estimate correlations between the different asset classes in his/her model.
- **Include Headers:** option for the user to include or discard a description on top of the different columns in the output files.

- **Interest Rate Outputs:** this option applies to nominal interest rates, to real interest rates and to term structures of risky bond prices. Bond prices will output prices of zero coupon bonds (i.e. zero-bond curves). **ContinuouslyCompoundedSpotRates** option will output spot rates with a continuous compounding frequency and **AnnuallyCompoundedSpotRates** option will output spot rates with a compounding frequency of one year.
- **Scenarios Optimization:** allows to apply numerical techniques with the aim of reducing simulation errors. The simulation of multiple factors with few scenarios requires a compromise between the wanted statistical properties.
- **Output Data Grouping:** the Linear option outputs term structures in rows and the projection periods in columns, while for the short rate, equity and inflation asset classes it outputs the simulations by columns. The Compact option outputs them the other way round.
- **Output Decimal Separator:** it defines whether the decimals are to be separated by commas or dots.
- **Output Format:** it defines whether the output files are to be in .csv or .txt format.
- **Measure:** it defines whether the scenarios are to be Risk Neutral or Real World.
- **Generate Risky Bond Prices:** tick this option to ensure that term structures for defaultable bonds are generated.
- **Risk Premium Calibration:** This is an option for the Real World simulations only. Choose **UseCustomValues** to manually insert your own view on risk premium values. Choose **UseMarketDataProviderValue** to obtain risk premium estimates from external data providers as Bloomberg and choose **UseFairmatCalibration** to let Fairmat automatically estimate these parameters with its built-in routines.
- **Custom Equity Market Price Of Risk:** Insert here your own view on what the market price of risk for the equity asset class should be.
- **Custom Inflation Market Price Of Risk:** Insert here your own view on what the market price of risk for the inflation should be.
- **Custom Inflation Market Price Of Risk:** Insert here your own view on what the market price of risk for the nominal rates should be.
- **Percentiles** This option applies to the test files. This option allows you to select the percentiles of simulated variables so you can visualize customized confidence intervals. The default values are 1%, 5%, 95% and 99% but any other percentile can be calculated in Fairmat.

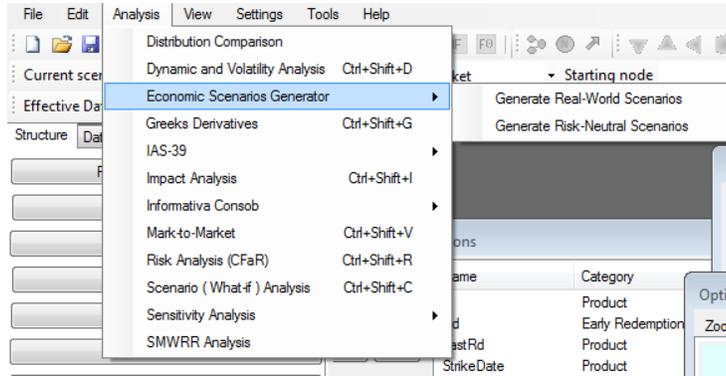


Figure 2: ESG Generation launch menu

## 4 Scenarios Generation

To generate scenarios follow the following steps.

- To specify the number of required scenarios go to **Settings/Numerical Settings** and set the field number of simulated paths.
- In order to start the generation of scenarios go to **Analysis/Economic Scenarios Generation** menu and selects either **Generate Real-World Scenarios** or **Generate Risk-Neutral Scenarios**.
- Please notes that depending on preferences, it may take several minutes to generate and save your scenarios. Furthermore Fairmat will gather market data from your market data provider if needed.

## 5 Simulators's Outputs

In this Section we will describe the outputs generated by the simulator and their relationships.

All the ESG scenarios are conveniently written in a series of output files for permitting post-processing with third parties actuarial tools. Every time the ESG generates a series of scenarios, a new folder will be created containing all the coherent scenarios. The folder name has a naming scheme like the following **ESG-2014-04-24-10-52-S2013-12-31-RN** in which is reported either the time stamp at which the simulation is required and the valuation date.

In the following subsections the content of each file will be described.

### 5.1 Equity files

Given  $S_t$  as the equity price, we will generate

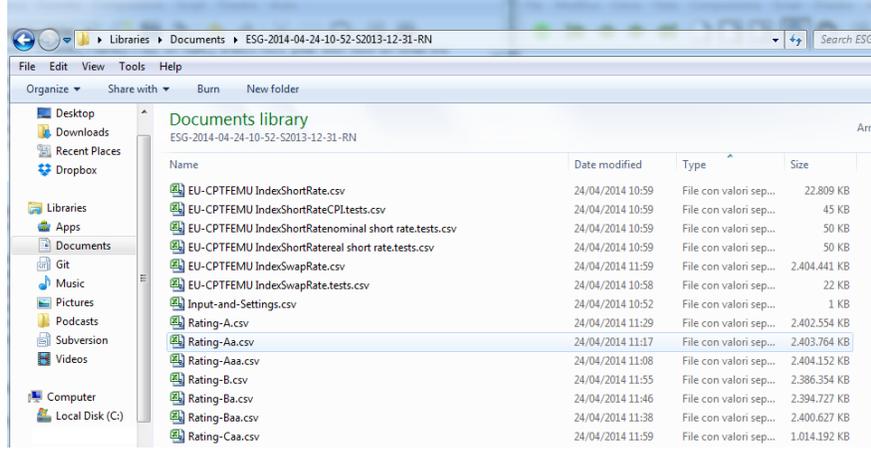


Figure 3: Example of ESG output folder.

- **Total Return:** the annualized return of the asset

$$TotalReturn_t = \frac{1}{dt} \left( \frac{S_t}{S_{t-dt}} - 1 \right)$$

where  $dt$  is the time step with which we are simulating. For example, if the time step is monthly,  $dt = 1/12$ . Equities are modeled as a geometric Brownian motion.

- **Dividend Yield  $\delta_t$ :** the dividend yield  $\delta_t$  is modeled by means of a mean-reverting process with a correlation  $\rho$  with the equity price process  $S_t$ . From the dividend yield, the one-period dividends are obtained as:

$$Dividends(t_1, t_2) = dt\delta_t S_t$$

Where  $dt$  is the time step with which we are simulating the processes, in our case, since we use monthly time steps,  $dt = 1/12$ .

Since the dividend yield process (and therefore the dividends) and the stock price process are modelled separately, we add them up in order to obtain the total return  $TR$  between two consecutive times  $t_1, t_2$ :

$$TotalReturn(t_1, t_2) = \frac{S_{t_2} + Dividends(t_1, t_2)}{S_{t_1}} - 1$$

- **Capital Change:** the capital change gives an idea of the appreciation or depreciation that the equity asset has experimented during a given period. The formula that relates it to the total return and the dividend yield is:

$$CapitalChange(t_1, t_2) = \frac{1 + TR(t_1, t_2)}{1 + \delta_{t_2} dt} - 1$$

## 5.2 Interest rates term structures

This section summaries several formulas that will be useful to switch from term structures generated in terms of the bond prices  $P(t, T)$  to the yield curves that clients will want as input to their actuarial systems.

Preliminar explanations:

$0$  is the current time, the origin of the term structure.  $T$  are the different maturities of the ZC bonds.  $t$  is a time between  $0$  and  $T$  at which we are interested in knowing the price of the ZC bond that will mature at time  $T$  (as seen from  $0$  but at time  $T - t$  as seen from  $t$ ).  $\tau(t, T)$  is the time (expressed in years) between the time instants  $t$  and  $T$ .

- Continuously compounded spot interest rates  $R(t, T)$  (see `ContinuouslyCompoundedSpotRates` option):

$$R(t, T) = \frac{-\ln P(t, T)}{\tau(t, T)}$$

- Annually compounded spot interest rate  $Y(t, T)$  (see `AnnuallyCompoundedSpotRates` option):

$$Y(t, T) = \frac{1}{P(t, T)^{\frac{1}{\tau(t, T)}}} - 1$$

## 5.3 Generated Scenarios Validation

Generated scenarios must be validated. Fairmat ESG will produce a series of output files (ending with the `.tests.csv` or the `.tests.txt` extension) containing several descriptive statistics and the non-arbitrage tests (for risk-neutral scenarios).

### 5.3.1 Non-Arbitrage test for equity models

Given  $r(z)$ , as the short-rate process  $R(s, t)$  the return rate for a given asset class, in the *one period returns test*, for every  $s, t$  with  $t > s$ , must hold:

$$E \left[ (1 + R(s, t)) \frac{e^{-\int_t^s r(z) dz}}{e^{-\int_0^s r(z) dz}} \right] = 1.$$

### 5.3.2 Non-Arbitrage test interest rates

As a way to double-check that the short rate model used is arbitrage free, the following test can be performed: Calculate the initial yield curve. This is given by the current term structure (the price of zero coupon bonds) If the short rate model is arbitrage free, we can calculate the zero coupon bond prices  $P(0, T)$  as

$$P(0, T) = E \left[ \frac{1}{e^{\int_0^T r(s) ds}} \right]$$

where this expectation is taken under the risk neutral measure, and the denominator represents the roll-up of the bank account.

From these  $P(0, T)$  estimated from the scenarios, one can build the yield curve. For example, if the yield curve is expressed in terms of the continuous compounded interest rate, the yield curve would be constructed as

$$R(0, T) = \frac{-\ln P(0, T)}{T}$$

If the yield curve constructed from the scenarios as described coincides with the initial yield curve we can be sure that the model for the short rate is arbitrage-free.

## 6 Theoretical Models and Numerical Techniques

This is a more technical section in which we will shortly review the theoretical models and techniques used for the simulator.

For a general reference on inflation indexed derivatives and the Jarrow-Yildirim model see [3] or chapter 16 and 17 of [1]. This plug-in contains also an enhanced version of the model in which the nominal rate which in the original model is calculated using the Hull-White model, is computed using the Pelsser Squared Gaussian model (See [5] and [4]) which have nicer properties with respect to the Hull-White model.

### 6.1 Jarrow-Yildirim process

- In the Fairmat user interface when you create a new stochastic process you will find the additional option “Jarrow-Yildirim”.
- The stochastic process is defined by the parameters shown in table below

Fairmat notation	Documentation notation
Nominal Zero Rate	$ZR_n(t)$
Real Zero Rate	$ZR_r(t)$
an	$a_n$
ar	$a_r$
Sigman	$\sigma_n$
Sigmar	$\sigma_r$
SigmaI	$\sigma_I$
rho_nI	$\rho_{nI}$
rho_nr	$\rho_{nr}$
rho_rI	$\rho_{rI}$
I0	$I_0$

where  $I_0$  is the starting value for the inflation index process while the others are parameters regulating the model dynamic.

- You can edit the model values shown in the table above directly on the “Parameters” tab of the Jarrow-Yildirim process window by double clicking on them.
- The first two inputs are functions so that you should specify a reference to a function in each of them.

### 6.1.1 Dynamics and interpretation of the parameters

The Jarrow-Yildirim model belongs to the set of inflation models based on the foreign-currency analogy.

The dynamics of the model in the risk neutral measure<sup>1</sup> are given by

$$dn(t) = [\theta_n(t) - a_n n(t)]dt + \sigma_n dW_n(t) \quad (1)$$

$$dr(t) = [\theta_r(t) - \rho_{rI}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dW_r(t) \quad (2)$$

$$dI(t) = I(t)[n(t) - r(t)]dt + \sigma_I I(t)dW_I(t) \quad (3)$$

where  $n(t)$  is the instantaneous nominal rate,  $r(t)$  is the instantaneous real rate and  $I(t)$  is the consumer price index (in the following CPI).

Nominal and real rate are modelled as two Hull-White processes. In analogy with that model the functions  $\theta_x(t)$  with  $x = n, r$  are calculated from starting term structures of nominal and real rates as

$$\theta_x(t) = \frac{\partial f_x(0, t)}{\partial t} + a_x f_x(0, t) + \frac{\sigma_x^2}{2a_x}(1 - e^{-2a_x t}) \quad , \quad x = n, r \quad (4)$$

where  $f_n(0, t)$  and  $f_r(0, t)$  are the nominal and real instantaneous forward rates at time 0 for maturity  $t$  and are observed in the market.

Parameters  $a_n$  and  $a_r$  can be interpreted as mean reversion speed while  $\sigma_n, \sigma_r, \sigma_I$  are volatilities of the processes. The three Wiener processes  $W_n, W_r, W_I$  are correlated through the correlation matrix

$$\rho = \begin{bmatrix} 1 & \rho_{nr} & \rho_{nI} \\ \rho_{nr} & 1 & \rho_{rI} \\ \rho_{nI} & \rho_{rI} & 1 \end{bmatrix}. \quad (5)$$

Note that only  $\rho_{rI}$  appears explicitly in the dynamics but the price of inflation indexed options will in general depend on all the three coefficients.

## 6.2 The Jarrow-Lando-Turnbull model for credit spreads

formulas

$$v^i(t, T) = P(t, T)[\delta + (1 - \delta)\tilde{Q}_t^i(\tau > T)]$$

<sup>1</sup>More precisely we should call it *nominal risk neutral measure*. We mean by this the measure that has the money market account based on the nominal rate as numeraire.

where  $v^i(t, T)$  is the price at time  $t$  of a defaultable bond of maturity  $T$ . Each superscript  $i$  corresponds with the credit rating classes being considered in the model.

$\delta$  is an exogenously given constant which represents the part of the face value of the bond that will be recuperated in case of default. It can be estimated from historical data and it can be specified for each rating class, however, for the sake of simplicity, in line with the original JLT paper (1997) we have assumed  $\delta=32.65\%$  for all rating classes.

$P(t, T)$  is the time  $t$  price of a riskless bond maturing at time  $T$ . JLT gives complete freedom to the user to specify whatever model for the riskless bond prices. In our case, we have opted for a Gaussian squared model developed by Antoon Pelsser. This model is similar to the usual one factor Hull-White model but it guarantees that interest rates will never get negative.

$\tilde{Q}_t(\tau > T)$  is the probability under the risk neutral measure that default occurs after date  $T$ .

From  $v^i(t, T)$  instantaneous forward rates for each rating class  $i$  are calculated as:

$$f^i(t, T) = -\frac{\partial \ln(v^i(t, T))}{\partial T}$$

The credit spread is defined as  $f^i(t, T) - f(t, T)$ , where  $f(t, T)$  is the riskless instantaneous forward rate, so plugging the JLT expression for  $v^i(t, T)$  and after some algebra one can get the following formula for the credit spread:

$$spd(t, T) = -\frac{(1 - \delta)\frac{\partial}{\partial T}\tilde{Q}_t^i(\tau > T)}{\delta + (1 - \delta)\tilde{Q}_t^i(\tau > T)}$$

### 6.2.1 Required market data and preprocessing

The input data that is necessary to the JLT model is: The one-year historical credit transition probability matrix. The risk-free zero-coupon bond prices curve. The risky zero-coupon bond prices curve for every credit rating category considered in the model. Recovery rate in the event of default of a bond.

One-year historical credit transition probability matrices are provided by rating agencies such as Moodys. For this case study we have used the credit transition probabilities matrix viewable in this internet webpage:

## 7 Real-World simulation

### 7.1 Equity processes under the real world measure

The key point for simulating equity process under the real-world measure is the determination of the equity premium. By following non-arbitrage theory[6] it may be stated that the equity (risk) premium is given by  $RP = \lambda\sigma$  where  $\lambda$  is the market price of risk.

## 7.2 Inflation under the real world measure

Risk premium for inflation is very small, currently we are using estimates available here [2].

## 7.3 Interest rates under the real world measure

Given the short-rate dynamics in the real-world measure expressed using the Hull-White model (but other models could be used)

$$r(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^P$$

and denoting with  $F^t(r(t), t)$  the price in time  $t$  of a zero coupon bond with maturity  $T$ , from Ito's lemma we can derive the price process as

$$dF^t = F^t \alpha^T dt + F^t \sigma^T dW^P$$

where

$$\alpha^T = \frac{F_t^T + \mu F_t^T + \frac{1}{2} \sigma^2 F_{rr}^T}{F^T}$$

$$\sigma^T = \frac{\sigma F_t^T}{F^T}$$

the subscripts indicate partial derivatives.

Imposing no arbitrage it's possible to find that two bonds with maturities  $T$  and  $S$  satisfy

$$\frac{\alpha^S(t) - r(t)}{\sigma^S(t)} = \frac{\alpha^T(t) - r(t)}{\sigma^T(t)}$$

From the fact that  $S$  appears only in the left hand side of the equality and  $T$  appears only in the right one it follows that they both are independent of the bond maturity letting us to define the market price of risk as

$$\lambda(t) = \frac{\alpha^T(t) - r(t)}{\sigma^T(t)}$$

It can be shown that in the risk-neutral measure the dynamics of the short-rate is

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma dW^Q$$

With the standard calibration of the short-rate model we determine the parameters entering in the risk-neutral dynamic and by knowing  $\lambda$  the real-world dynamics can be determined. The problem of simulating the short-rate in the real-world measure is thus reduced to that of estimating  $\lambda$ .

The simplest possible assumption on  $\lambda$  is clearly to assume it constant. Its value can then be estimated from the historical series of zero rate curves: given a historical series of zero rate curves it's possible to calculate the series of prices

of a zero coupon bond maturing at a certain time, let's indicate this series with  $F^T(i)$ . After calculating the series of returns

$$R_i^T = \frac{F^T(i+1)}{F^T(i)} - 1$$

the two parameters  $\alpha^T$  and  $\sigma^T$  can be estimated from its mean and standard deviation as follows:

$$\alpha^T = \frac{\text{Mean}(R_i)}{dt}$$

and

$$\sigma^T = \frac{\text{StandardDeviation}(R_i)}{dt}$$

where dt represents the interval between observations. From the zero rate series it's also possible to calculate a proxy for the historical series of short rate by extrapolating the curve at zero for each observation. Taking the mean value for the short rate, let's call it r, a rough estimate for lambda is given by

$$\lambda = \frac{\alpha^T - r}{\sigma^T}$$

This procedure can be carried on for different bond maturities returning different estimates for lambda.

## 7.4 Variance reduction techniques

It is a well-known fact that, in order to get accurate estimates through the (crude) Monte Carlo method, one needs to use a big number of simulations since the estimation error is proportional to  $1/\sqrt{(N)}$ , where N is the number of simulations. In practice, to obtain an extra decimal of precision in your Monte Carlo estimate, it is needed to increase the number of simulations 100 times (i.e., if you were doing N=1,000 simulations, you need to increase the number of simulations to N=100,000.)

To increase accuracy and/or speed of simulation there are available a number of techniques that are collectively called Variance reduction techniques.

Within Fairmat ESG, we have implemented the following techniques:

- **Moment Matching.** The moment matching technique consists in adjusting the simulated data so these data have sample moments that are equal to the theoretical population moments of the random variable of interest. For example, if we were simulating a GBM, we would adjust the simulated paths so at the terminal time the stock price's mean and variance would be the theoretically known mean and sample, so in this way we have a sample free from sampling error.
- **Antithetic Variates:** take advantage of the of the symmetry of input distribution.

## References

- [1] Damiano Brigo and Fabio Mercurio. *Interest Rate Models, Theory and Practice*. Springer, second edition, 2006.
- [2] Thomas Werner Juan Angel García. <http://www.ecb.europa.eu/pub/pdf/scpwps/ecbwp1162.pdf>.
- [3] Fabio Mercurio. Pricing inflation-indexed derivatives. *Quantitative Finance*, 5(3):289–302, 2005.
- [4] Antoon Pelsser. A Tractable Yield-Curve Model That Guarantees Positive Interest Rates. *Review of Derivatives Research*, (1):269–284, 1997.
- [5] Antoon Pelsser. *Efficient Methods for Valuing Interest Rate Derivatives*. Springer Finance, 2000.
- [6] Stephen A. Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360, 1976.