# Variance Gamma Model
# Version 1.0

## 1 Preface

*This plug-is has been developed starting from the University of Edinburgh MSC in finance and operation research dissertation of Safe Khampol. The aim of the dissertation was to implement a calibration and Monte Carlo simulation procedure for The Variance Gamma (VG) Model for Matlab and for the Fairmat plug-in system.*

## 2 Introduction

The Variance Gamma (VG) is a mathematical model for simulating stock prices and allows closed form calculation of European options, one popular financial derivative product. The model is both simple and robust at the same time and is a very good alternation to the existing models like Heston stochastic volatility model and Black-Scholes Model which may be biased when applied to real market circumstance. (Comment:Heston is good no comparison with Heston unless)

In the codes, we generated entirely option prices of VG examples by picking all the input variables and using VG closed form and VG Fast Fourier Transform methods to compute their values. After that, each method that has been used in the example will be served as model price in the calibration. Then, the VG calibration will estimate three parameters $\theta$, $\sigma$ and $\upsilon$ and those parameters will be substituted into computation of VG Monte Carlo and Back Scholes option pricing and VG Simulation of Price Path as Gamma time-changed Brownian Motion and difference of Gamma.

As a reminder, all the symbols or names in C# and Matlab have the same meaning by assuming $s_0$ is the current stock price, k is strike price, t (or T-t) is time to maturity starting from time 0, q is a dividend, $\theta$ (theta) is the drift in the Brownian motion, $\sigma$ (sigma) is the volatility of the Brownian motion, and $\upsilon$ (nu) is the variance rate of gamma time change. Some methods do not include the dividend, q.

In this work we implemented both valuation models of Black-Scholes and Variance Gamma. In VG model, there are various ways to find VG option price. However, the recommended VG option pricing models are VG closed form, VG Monte Carlo, Fast Fourier Transform methods and will be explained in details. In case of the calibration of parameters, either Fast Fourier transform or closed-form has different advantages such as time of calibration, accuracy of estimated parameters and limitation of constraint. Finally, the VG simulation of stock price and comparison of BS and VG option are employed by calibrated parameters. These topics will be explained in the following sections.

# 3 Variance Gamma Closed-form formulas for European Call options

The first formula is closed-form for Variance Gamma European option pricing model by Madan, Carr and Eric. The original equation contradicts with the following formula because it contains mistakes in the model. The errors were in the way of computation of $\mathrm{a}_1$, $\mathrm{b}_1$, $\mathrm{a}_2$, and $\mathrm{b}_2$.This is also confirmed from Professor Peter P. Carr e-mail, but he didn't point out where exactly. However, the below formula was implemented by Whitley (2009) and I has been re-checked its solution with other Variance Gamma methods.

Its closed-form is originated from the Classic Black-Scholes model and adapted to the conditional normality of the process and the Variance Gamma process as drift of Brownian motion with time span from a Gamma distribution and the risk neutral dynamic of the stock price. As a result, it uses a high computation process, due to the modified Bessel function of the second kind and integration of the degenerate hyper-geometric function of two variables, and is given as

$$\mathrm{Call}\ (\mathrm{s}_0,\ \mathrm{k},\ \mathrm{t}) = \mathrm{s}_0\, e^{-rt}\psi\ (\mathrm{a}_1,\ \mathrm{b}_1,\ \gamma) - \mathrm{k}\ e^{-rt}\quad \psi\ (\mathrm{a}_2,\ \mathrm{b}_2,\ \gamma)$$

where

$$\gamma \quad = \frac{t}{\upsilon}$$

$$\omega = \frac{1}{\upsilon}\ln?(1 - \upsilon\theta - \frac{1}{2}\sigma^2\upsilon)$$

$$\zeta = \frac{(\ln\left(\frac{s_t}{k}\right) + \omega t)}{\sigma}$$

$$\vartheta = 1 - \upsilon\,(\theta + \frac{1}{2}\sigma^2)$$

$$a_1 \;= \zeta\sqrt{\frac{\vartheta}{\upsilon}}$$

$$b_1 \;= \frac{1}{\sigma}(\theta + \sigma^2)\,\sqrt{\frac{\upsilon}{\vartheta}}$$

$$a_2 = \zeta\,\sqrt{\frac{1}{\upsilon}}$$

$$b_2 = \frac{1}{\sigma}\,\theta\sqrt{\upsilon}$$

The function $\psi$ is defined in terms of the modified Bessel function of the second kind, $K_v(z)$, and the degenerate hyper-geometric function of two variables, $\Phi$ has the integral representation Humbert (1920), as

$$\psi\ (\mathrm{a},\ \mathrm{b},\ \gamma) = \frac{c^{\gamma+\frac{1}{2}} e^{sign(a)c}\,(1+u)^{\gamma}}{\sqrt{2\pi}\,?\,(\gamma)\,\gamma} K_{\gamma+\frac{1}{2}}(c)\Phi\left(\gamma,\ 1-\gamma,\ 1+\gamma;\ \frac{1+u}{2},\ -\mathrm{sign}\,(\mathrm{a})\,\mathrm{c}\ (1+\mathrm{u})\right)$$

$$-\mathrm{sign}(\mathrm{a})\ \frac{c^{\gamma+\frac{1}{2}} e^{sign(a)c}\,(1+u)^{1+\gamma}}{\sqrt{2\pi}\,?\,(\gamma)\,(1+\gamma)}\ K_{\gamma-\frac{1}{2}}\,(c)\,\Phi\left(1+\gamma,\ 1-\gamma,\ 2+\gamma;\ \frac{1+u}{2},\ -sign\,(a)\,c\,(1+u)\right)$$

$$+sign(a)\frac{c^{\gamma+\frac{1}{2}} e^{sign(a)c}\,(1+u)^{1+\gamma}}{\sqrt{2\pi}?\,(\gamma)\,\gamma} K_{\gamma-\frac{1}{2}}(c)\Phi\left(\gamma,\ 1-\gamma,\ 1+\gamma;\ \frac{1+u}{2},\ -sign\,(a)\,c(1+u)\right),$$

where c = $|a|\sqrt{2+b^2}$, $u = \frac{b}{\sqrt{2+b^2}}$ and

$$\Phi\ (\alpha,\ \beta,\ \gamma,\ x,\ y) = \frac{?(\gamma)}{?(\alpha)?(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}e^{uy}\,\partial u$$

where $?(x)$ is the gamma function.

However, the degenerate hyper geometric function of two variables, $\Phi$, has a strong singularity. As a result, in Matlab the adaptive Simpson quadrature integration will be the most effective method because at least there will be solutions for any given parameters. On the other hands, in C#, regarding this problem, I adapted Humbert series expansion for confluent Hypergeometric function of two variables and wrote a C# for it. The Confluent hypergeometric functions of two variables have the following integral form:

$$\Phi(\alpha,\beta,\gamma,x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int\limits_0^1 v^{\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx)^{-\beta}e^{vy}dv$$

and the series form:

$$\Phi(\alpha,\beta,\gamma,x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^m y^n, |x|<1, |y|<1$$

where the Pochhammer symbol (q)$_n$ denotes the rising factorial:

$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)} = q(q+1)\cdots(q+n-1)$$

This can imply the series is still convergent with |x|<1and without condition |y|<1 because

$$x = \frac{1+u}{2} = \frac{1+\frac{b}{\sqrt{2+b^2}}}{2} \ \Rightarrow\ |x| \le \frac{1}{2} + \frac{|b|}{2\sqrt{2+b^2}} = \frac{1}{2} + \frac{|b|}{2\sqrt{2+b^2}} < 1,\ \ \forall b$$

means |x|<1 in this model. Formula for my integral:

$$I(\alpha,x,y)=\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\Phi(\alpha,1-\alpha,1+\alpha,x,y)=\int\limits_0^1 v^{\alpha-1}(1-vx)^{\alpha-1}e^{vy}dv,\ \ 0<\alpha,x,y<1$$

Another way around, it also can use an expansion related to Gauss hypergeometric function (${}_2F_1(a,b;c;x)$ is Gauss hyper-geometric function and can be calculated with Matlab function hypergeom), shown as below.

$$\begin{aligned}
&\Phi(\alpha,\beta,\gamma;x,y)=\tfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int\limits_0^1 v^{\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx)^{-\beta}e^{vy}dv\\
&=\tfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\sum_{n=0}^{\infty}\tfrac{y^n}{n!}\int\limits_0^1 v^{n+\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx)^{-\beta}dv\\
&=\sum_{n=0}^{\infty}\tfrac{\Gamma(\gamma)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\gamma)}\,{}_2\tfrac{y^n}{n!}F_1(\beta,n+\alpha;n+\gamma;x),\quad |x|<1.
\end{aligned}$$

In conclusion, these formulas should work in theory, but there is an extreme result and rarely occurs wrong answers. Both forms produce the same answer.

# 4 Variance Gamma Monte Carlo Simulation for European Call options

A Monte Carlo simulation is a procedure for sampling random outcomes for the applying process, VG process. It helped to evaluate and analyze investments by simulating the various sources if uncertainty affecting their value, and then determining their average value of sampling outcomes. Its option pricing way is straightforward: sample a sufficiently big number of stock price paths of the VG process to obtain VG stock price paths under the risk neutral measure between time 0 and t. The initial inputs are time at maturity t, skewness $\theta$, volatility $\sigma$, variance rate of the gamma process $v$, risk-free rate r, dividend q,

, and spot price s0. $\omega$ is the same equation as before, $\frac{1}{v}\ln\left(1-\theta v-\frac{\sigma^2 v}{2}\right)$

The sampling g values are generated by using gamma inverse cumulative distribution function. The inverse of the gamma cumulative distribution function with shape parameters in t / $v$ and scale parameters in $v$ for the corresponding probabilities in random number from uniform distribution of zero mean and unit variance between 0 and 1.

$$g=gammainv(uniform\,(0,1)\,,\,\mathrm{t}\,/\,v,\,v\,)$$

Each value of g sample is normally distributed with mean $\theta$g and standard deviation $\sigma\sqrt{g}$ with inverse standard normal distribution function. Sometimes, g is referred to "measuring economic time or time adjusted for the flow of information" as well as t is "the usual time measure".

$$h=\theta g+\sigma\sqrt{g}\,norminv(uniform\,(0,1))$$

The stock price s$_t$ is given by

$$s_t = s_0 exp\left[\left(r - q\right)t + \omega t + h\right]$$

Then, the European call price is

$$c\left(s_0; k, t\right) = \, e^{-rt} E\left[max\left(s_t - k, 0\right)\right]$$

Thus, the variance gamma call and put option VG Monte Caro simulations are

$$c = \, e^{-rt} \frac{\sum\limits_{i=1}^{n} max\left[s_t\left(i\right) - k, 0\right]}{n}$$

$$p = \, e^{-rt} \frac{\sum\limits_{i=1}^{n} max\left[k - s_t\left(i\right), 0\right]}{n}$$

where n is number of simulation of stock price.

Another way around of characterizing gamma distribution is that the arriving rate of information during time t is defined by g. If g is large, large arriving information and the sample of stock price is a normal distribution in “h” from above equation which has a relatively large mean and variance. Conversely, if g is small, a small deal of information arrives and the sample of stock price has a relatively small mean and variance.

If $\upsilon$ is getting closer to zero in the variance gamma model, the value of g will be becoming T with certainty and can be illustrated by using the gamma distribution function. In addition, $\omega$ is –$\theta$t from using a series expansion for the stock price function; consequently, in the limit the distribution of S$_t$ has a mean of $S_0 \exp[(r - q)t]$ and a standard deviation of $\sigma\sqrt{t}$ then this model converts to geometric Brownian motion.

### 4.1 Calculation of Variance Gamma Put for European options

The value of the European put is calculated from the value of the European call option by put-call parity, as

$$\text{Put } (s_t, \text{ k, t}) = \text{Call } (s_t, \text{ k, t}) \text{ - } s_t \text{ e}^{-qt} \quad + \text{ k e}^{-rt}$$

## 5 Calibration of VG parameters from its formula

Madan, Carr and Chang (The Variance Gamma Process and Option Pricing, 1998) also introduced the tuned variance gamma process in the index options market and statistical process to the level of the fourth moment. There are varying numbers of option prices range and thus, it is typically impossible to identify a single set of parameters that would fit right to all options prices. As a result, maximum likelihood method is employed to estimate these parameters. To be exact, the likelihood employed addresses expected heteroskedasticity in option prices for various strikes point by adopting a multiplicative error formulation. The maximum likelihood estimation is asymptotically equivalent to the minimization of followed, with $\omega_i$ as the observed market price on the i$^{th}$ and $\hat{\omega}_i$ as the model price.

$$k = \sqrt{\frac{1}{M}\sum i = 1^M[\ln(\omega_i) - ln(\hat{\omega}_i)]}$$

These include the average estimated parameter values of the risk neutral densities and its accompanying standard deviation throughout the observation, and also the minimum and maximum estimated value. After the research of model in Matlab, the variance gamma option pricing model, $\hat{\omega}_i$, in implementation will be used only two methods: variance gamma Closed Form and Fast Fourier transform because they are given a reasonable parameters and speed of evaluating computation. In contrast, VG Monte Carlo takes a long time to calibrate the parameters.

# 6 Simulation of Variance Gamma Price Paths

Those calibrating three parameters and given parameters will be substituted to find variance gamma price path. There are three main methods to simulate VG. However, in this paper, two methods are only considered and implemented and the third one for simulating VG is approximated by a compound Poisson process. The two models based on two representations presented before and have closely and correctly distribution. The only advantage of the third method that the other two cannot do is that it can be used for any Lévy process.

In the following figure, the two different algorithms for sequentially generating VG sample paths between 0 and T starting from 0 (0 = t$_0$ <t$_1$ <. . . <t$_N$ = T) where the time increments $\Delta t_i, i = 1, ..., N$ are given as input as well as VG parameters. **Figure 1: Simulating VG stock price as Gamma Time-Changed Brownian motion**

Input: VG parameters ($\theta$, $\sigma$, $\upsilon$); time spacing $\Delta t_1, \ldots, \Delta t_N$ subjected to $\sum_{i=1}^{N} \Delta t_i = T$

Initialization: Set X$_0$ = 0.

Loop from i = 1 to N:

1. Generate $\Delta G_i \sim ?\left(\frac{\Delta t_i}{\upsilon}, \upsilon\right) and\, normal\, Z_i \sim N\,(0,1)$

independent and independent of past random variates

1. Return $X\,(t_i) = X\,(t_{i-1}) + \theta \Delta G_i + \sigma\sqrt{\Delta G_i} Z_i$.
2. Computing stock price: $S\,(t_i) = S\,(0)\exp?\,((r+\omega)\,t + X\,(t_i))$

**Figure 2: Simulating VG stock price as Difference of Gamma**

Input: VG parameters $(\theta,\ \sigma,\ \upsilon)$; time spacing $\Delta t_1, \ldots, \Delta t_N$ subjected to $\sum_{i=1}^{N} \Delta t_i = \ T$

Initialization: Set X$_0$ = 0.

Loop from i = 1 to N:

1. Generate independent gamma variates $\Delta\gamma_i^- \sim ?\left(\frac{\Delta t_i}{\upsilon},\ \upsilon\mu_n\right)$, $\Delta\gamma_i^+ \sim ?\left(\frac{\Delta t_i}{\upsilon},\ \upsilon\mu_p\right)$, independently of past random variates
2. Return $X\left(t_i\right) = X\left(t_{i-1}\right) + \Delta\gamma_i^+ - \Delta\gamma_i^-$
3. Computing stock price: $S\left(t_i\right) = S\left(0\right) \exp?\left(\left(r + \omega\right) t + X\left(t_i\right)\right)$